A Technique for Computing the Convolution of Exponential Signals and its Application in Systems Theory

Francisco Mota
Dept. of Computer Engineering,
Univ. Fed. do Rio G. do Norte, Natal, RN – Brasil.
Email: mota@dca.ufrn.br

Tania Luna
Dept. of Electrical Engineering,
Univ. Fed. Rural do Semi-Arido, Caraubas, RN – Brasil.
Email: tania.luna@ufersa.edu.br

***Abstract*—We present a procedure for computing the convolution of (analog-time) exponential signals without the need of solving integrals. The procedure is algebraic and requires the resolution of a system of linear equations. The method can be applied to find the impulse response of a linear time-invariant (LTI) system modeled by an ordinary differential equation and also to compute the solution of an ordinary differential equation with constant coefficients. We believe this technique can be useful for the analysis of LTI systems in time domain, since it is conceptually simple and can be easily implemented as algorithms in software packages like Scilab and/or Matlab.**

Keywords: Exponential Signal, Convolution, Vandermonde Matrix, Linear Time-Invariant (LTI) System, Ordinary Differential Equation.

I. Introduction

Convolution between signals [1] is a fundamental operation in the context of the theory of linear time invariant (LTI) systems and its importance comes mainly from the fact that this operation is linear on the signals and has the property of commuting with translations and differentiations. These properties imply the following pivotal result in theory of systems: *The output signal of a LTI system is given by the convolution between the input signal and the system impulse response signal*. So, the impulse response completely characterizes a LTI system for the purpose of input-output analysis. For LTI systems whose model of ordinary differential equation (ODE) is known, we have the important result that *"the system impulse response can be decomposed as a convolution of exponential signals"*, which can be obtained from the model [2]. These facts implies that the computation of convolution involving exponential signals is an important question in the context of linear systems theory and we consider it deserves a more deep analysis than is generally presented in basic textbooks on theory of linear systems. As we show bellow, convolution between exponential signals have some properties that can be explored to devise algorithms for obtaing a LTI impulse response as well as for solving ODEs in a very simple way. Due to its simplicity we believe that this approach may be teached even in a first course on linear systems theory.

A. Notation and Definitions

The notation we use is quite standard and it is presented here only for the sake of self-containdness:

- $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are, respectively, the set of integers, real and complex numbers;
- The binomial coefficient is defined for $n, p \geq 0 \in \mathbb{Z}$ as:
$$\binom{n}{p} = \begin{cases} 0, & n < p \\ \dfrac{n!}{(n-p)!p!}, & n \geq p \end{cases}$$
- An (analog-time) signal is defined as a complex valued function $f : \mathbb{R} \to \mathbb{C}$. In this paper we are mainly concerned with exponential signals, that is, $f(t) = e^{rt}$ where $r \in \mathbb{C}$. Two basic signals will be necessary in our development, namely, the unit step signal (σ) and the unit impulse (generalized) signal[1] (δ). The unit step σ is defined as
$$\sigma(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$
and the unit impulse can be defined as the derivative of σ, that is, $\delta = \dot{\sigma}$. If f is a signal, the product "σf" is given by
$$(\sigma f)(t) = \sigma(t) f(t) = \begin{cases} 0, & t < 0 \\ f(t), & t > 0 \end{cases}$$
If f is continuous at $t = 0$ we can also compute the product "δf" as $\delta f = f(0)\delta$. Additionally, if the signal f is differentiable we can obtain the derivative of "σf" as shown bellow:
$$(\dot{\sigma f}) = \dot{\sigma} f + \sigma \dot{f} = \delta f + \sigma \dot{f} = f(0)\delta + \sigma \dot{f},$$
which will be useful in proof of Theorem 1.
- The convolution between two signals f and g, represented by $f * g$, is the binary operation defined as (see [1]):
$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau \qquad (1)$$

[1] A generalized signal is in fact a distribution, as defined in [4], [5].

Additionally, if $f(t) = g(t) = 0$ for $t < 0$, we get from (1) that:

$$(f * g)(t) = \begin{cases} 0, & t < 0 \\ \displaystyle\int_0^t f(\tau)g(t-\tau)d\tau, & t > 0 \end{cases} \tag{2}$$

Convolution is commutative, associative and the unity of the operation is the (generalized) signal δ, that is, $f * \delta = \delta * f = f$ for any signal f.[2] Another important property of convolution is related with signal derivative:

$$(\dot{f * g}) = \dot{f} * g = f * \dot{g} \tag{3}$$

II. Convolution between exponential signals

In this section we consider signals $h : \mathbb{R} \to \mathbb{C}$ defined by:

$$h(t) = \begin{cases} 0, & t < 0 \\ e^{rt}, & t > 0 \end{cases} \quad r \in \mathbb{C}, \tag{4}$$

or, alternatively, $h(t) = \sigma(t)e^{rt}$, where σ is the unit step signal.

Theorem 1. Consider the convolution of $n \geq 2$ signals $\{h_1, h_2, \ldots, h_n\}$ with $h_j(t) = \sigma(t)e^{r_j t}$, where σ is the unit step signal and $r_j \in \mathbb{C}$. The i-th derivative of the convolution $(h_1 * h_2 * \cdots * h_n)$, represented by $(h_1 * h_2 * \cdots * h_n)^{(i)}$, evaluated at $t = 0^+$ is given by:

$$(h_1 * h_2 * \cdots * h_n)^{(i)}(0^+) = \begin{cases} 0, & i = 0, 1, \ldots, n-2 \\ 1, & i = n-1 \end{cases} \tag{5}$$

and we consider $(h_1 * h_2 * \cdots * h_n)^{(0)} = h_1 * h_2 * \cdots * h_n$.

Proof. See Appendix □

In the following we will find a procedure for computing the convolution $h_1 * h_2 * \cdots * h_n$ for $n \geq 2$ and $h_j(t) = \sigma(t)e^{r_j t}$ with $r_j \in \mathbb{C}$ without the need of solving integrals. To begin with, we consider the case where $h_i \neq h_j$ for $i \neq j$, which implies $r_i \neq r_j$ for $i \neq j$.

Theorem 2. The convolution between $n \geq 2$ exponentials signals $\{h_1, h_2, \ldots, h_n\}$, with $h_j(t) = \sigma(t)e^{r_j t}$, $r_j \in \mathbb{C}$ and $h_i \neq h_j$ for $i \neq j$, is given by

$$h_1 * h_2 * \cdots * h_n = A_1 h_1 + A_2 h_2 + \cdots + A_n h_n, \tag{6}$$

where $A_j \in \mathbb{C}$ are scalars that can be computed by solving a linear system $VA = B$ where V is the $n \times n$ (non-singular) Vandermonde matrix defined by $V_{ij} = r_j^{i-1}$, A and B are the n-column vectors $A = (A_1, A_2, \ldots, A_n)$ and $B = (0, 0, \ldots, 1)$, that is:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{7}$$

[2] generalized signal δ can be convoluted naturally with any signal as well as with itself.

So, vector A is the last (n-th) column of the inverse of V.

Proof. We use induction on n to prove (6), which is easily verified for $n = 2$ using definition of convolution in Equation (2). Supposing (6) valid for $n = k \geq 2$, and using the fact that convolution is associative we easily prove (6) for $n = k + 1$. To obtain (7) we take the i-th derivative at $t = 0^+$ on both sides of (6) to get:

$$(h_1 * \cdots * h_n)^{(i)}(0^+) = A_1 h_1^{(i)}(0^+) + \cdots + A_n h_n^{(i)}(0^+), \quad i = 0, 1, 2, \ldots, n-1.$$

Applying Theorem 1 to left side of equation above and using the fact that $h_j^{(i)}(0^+) = r_j^i$ we get (7).

□

Now we consider the more general convolution $h_1 * h_2 * \cdots * h_n$, $n \geq 2$, where there is the possibility of some h_i to be repeated, that is $h_i = h_j$ for some $i \neq j$. We initially consider some facts about the so-called "n-fold convolution" (or "convolution power" [6], [7]) of exponentials, that is, the convolution of h, as defined in (4), repeated between itself n times, and we represent it by h^{*n}.

Lemma 2.1. The n-fold convolution of $h(t) = \sigma(t)e^{rt}$, is given by

$$h^{*n}(t) = \underbrace{(h * h * \cdots * h)}_{n \text{ terms}}(t) = \frac{1}{(n-1)!}t^{n-1}h(t), \quad n \geq 1 \tag{8}$$

Proof. By induction on n. It is trivially true for $n = 1$ and supposing it valid for $n = k$ we can prove for $n = k + 1$ by writing $h^{*(k+1)} = h^{*k} * h$ and using definition of convolution in Equation (2). □

The Lemma bellow shows a version of Theorem 1 applied to the n-fold convolution of h:

Lemma 2.2. Let be $h(t) = \sigma(t)e^{rt}$, then i-th derivative of $h^{*n}(t)$, $n \geq 1$, computed at $t = 0^+$ is given by:

$$(h^{*n})^{(i)}(0^+) = \binom{i}{n-1} r^{i-n+1} \tag{9}$$

Proof. Equation (9) follows from Lemma 2.1 by setting $k = n - 1$ in the well-known formula bellow:

$$\left.\frac{d^i}{dt^i}\left(\frac{t^k}{k!}e^{rt}\right)\right|_{t=0} = \binom{i}{k} r^{i-k}$$

□

Now we analyse how it would be like the convolution $h_1^{*n_1} * h_2^{*n_2}$, where $h_1(t) = \sigma(t)e^{r_1 t}$ and $h_2(t) = \sigma(t)e^{r_2 t}$, with $r_1 \neq r_2$, that is the convolution between the "n_1-fold" convolution of h_1 with the "n_2-fold" convolution of h_2 when $h_1 \neq h_2$.

Lemma 2.3. Let be $h_1(t) = \sigma(t)e^{r_1 t}$ and $h_2(t) = \sigma(t)e^{r_2 t}$, with $r_1 \neq r_2$, the convolution between the n_1-fold convolution of h_1 and the n_2-fold convolution of h_2, denoted by $h_1^{*n_1} * h_2^{*n_2}$, is given by:

$$\begin{aligned} h_1^{*n_1} * h_2^{*n_2} &= \underbrace{(h_1 * h_1 * \cdots * h_1)}_{n_1 \text{ terms}} * \underbrace{(h_2 * h_2 * \cdots * h_2)}_{n_2 \text{ terms}} \\ &= (A_1 h_1 + A_2 h_1^{*2} + \cdots + A_{n_1} h_1^{*n_1}) + \\ &\quad + (B_1 h_2 + B_2 h_2^{*2} + \cdots + B_{n_2} h_2^{*n_2}) \end{aligned}$$

Proof. See Appendix. □

Bellow we present a general result about the power convolution of n exponential signals as show in (4) which is a generalization of Theorem 2:

Theorem 3. The convolution between $n \geq 2$ exponential signals $\{h_1, h_2, \ldots, h_n\}$, with $h_i(t) = \sigma(t)e^{r_i t}$, $r_i \in \mathbb{C}$ and q distinct h_s, each of them repeated n_s times, so that $n_1 + n_2 + \cdots + n_q = n$, is given by

$$\begin{aligned} h_1^{*n_1} * h_2^{*n_2} * \cdots * h_q^{*n_q} &= \sum_{j=1}^{n_1} A_{1j} h_1^{*j} + \sum_{j=1}^{n_2} A_{2j} h_2^{*j} + \cdots \\ &\quad + \sum_{j=1}^{n_q} A_{qj} h_q^{*j}. \end{aligned} \tag{10}$$

Alternativelly, by using (8) in Lemma 2.1 we have $h_i^{*j}(t) = \frac{1}{(j-1)!} t^{j-1} h_i(t)$, and then we can rewrite (10) as:

$$h_1^{*n_1} * h_2^{*n_2} * \cdots * h_q^{*n_q} = p_1 h_1 + p_2 h_2 + \cdots + p_q h_q \tag{11}$$

where each p_s, $s = 1, \ldots, q$, is a polynomial defined as

$$p_s(t) = \sum_{j=1}^{n_s} A_{sj} \frac{t^{j-1}}{(j-1)!}$$

and constants $A_{sj} \in \mathbb{C}$ are scalars that can be computed by solving a linear system $VA = B$ where V is the $n \times n$ nonsingular confluent (or generalized) Vandermonde matrix defined by $V = \begin{bmatrix} V_1 & V_2 & \cdots & V_q \end{bmatrix}$, where each block V_s is the $n \times n_s$ matrix whose entries are defined by

$$(V_s)_{ij} = \binom{i-1}{j-1} r_s^{i-j},$$

A and B are the n-column vectors $A = (A_1, A_2, \ldots, A_q)$, each A_s is a n_s-column vector, and $B = (0_1, 0_2, \ldots, B_q)$, where 0_s are n_s-column zero vectors and B_q is the n_q-column vector $(0, 0, \cdots, 1)$ that is:

$$\begin{bmatrix} V_1 & V_2 & \cdots & V_q \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_q \end{bmatrix} = \begin{bmatrix} 0_1 \\ 0_2 \\ 0_3 \\ \vdots \\ B_q \end{bmatrix} \tag{12}$$

So, vector A is the last (n-th) column of the inverse of V.

Proof. See Appendix. □

III. Analysis of LTI Systems Modeled by Ordinary Differential Equations

Consider a causal LTI system modelled by the following ordinary differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = u, \quad a_i \in \mathbb{R} \tag{13}$$

where u is the input signal and y is the output signal. The impulse response (h) for this system is given by the convolution:

$$h = h_1 * h_2 * \cdots * h_n, \quad h_i(t) = \sigma(t) e^{r_i t}, \quad r_i \in \mathbb{C}$$

and $r_1, r_2, \ldots, r_n$ are the roots of the characteristic equation $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. Supposing that the characteristic equation has q distinct roots r_s, each one repeated n_s times, so that $n_1 + n_2 + \cdots + n_q = n$, then we can obtain the impulse response h by using Theorem 3, Equation (11), that is

$$h = p_1 h_1 + p_2 h_2 + \cdots + p_q h_q \tag{14}$$

where $h_s(t) = \sigma(t) e^{r_s t}$, $p_s(t) = \sum_{j=1}^{n_s} A_{sj} \frac{t^{j-1}}{(j-1)!}$, and A_{sj}, $s = 1, \ldots, q$ and $j = 1, \ldots, n_s$, are calculated by solving the Vandermonde system (12).

On the othe hand, it is well known that the complete solution of differential equation (13) is generally written as

$$y = y_h + y_p \tag{15}$$

where y_h is the homogeneous (or zero input) solution and y_p is a particular solution, i.e., it depends on input signal u. When solving (13) for $t \geq 0$, the particular solution y_p can be written as

$$y_p(t) = \int_0^t u(\tau) h(t-\tau) d\tau = [(\sigma u) * h](t), \tag{16}$$

$$\text{where } (\sigma u)(t) = \begin{cases} 0, & t < 0 \\ u(t), & t > 0 \end{cases}$$

and h is the impulse response. The homogeneous solution (y_h) has the same format of (14), that is

$$y_h = \bar{p}_1 h_1 + \bar{p}_2 h_2 + \cdots + \bar{p}_q h_q, \tag{17}$$

$$\text{where } h_s(t) = e^{r_s t}, \text{ and } \bar{p}_s(t) = \sum_{j=1}^{n_s} \bar{A}_{sj} \frac{t^{j-1}}{(j-1)!}.$$

Therefore to solve (13) we need to obtain y_p and y_h. To find y_h we use the fact that the particular solution y_p, as shown in (16), is a convolution between $n+1$ signals, namely, "$y_p = (\sigma u) * h = (\sigma u) * h_1 * h_2 * \cdots * h_n$", and conclude, by using Theorem 1, that:

$$y_p(0^+) = \dot{y}_p(0^+) = \ddot{y}_p(0^+) = \cdots = y_p^{(n-1)}(0^+) = 0$$

and so, using these conditions in (15), we get:

$$y(0^+) = y_h(0^+),\ \dot{y}(0^+) = \dot{y}_h(0^+),\ \ddot{y}(0^+) = \ddot{y}_h(0^+), \cdots, \\ y^{(n-1)}(0^+) = y_h^{(n-1)}(0^+).$$

This set of conditions on y_h can be used to find the constants $\bar{A}_{sj}$ in (17) since the "initial values"

$y(0), \dot{y}(0), \ddot{y}(0), \ldots, y^{(n-1)}(0)$ are generally known when solving (13) for $t \geq 0$. This implies that the constants $\bar{A}_{sj}$, $s = 1, \ldots q$ and $j = 1, \ldots, n_s$, can be computed by solving a Vandermonde system like the one showed in Theorem 3, that is $V\bar{A} = \bar{B}$, where the Vandermonde matrix V is the same one used to compute the impulse response h, $\bar{A}$ is the $n \times 1$ vector composed by the $\bar{A}_{sj}$'s and the vector $\bar{B}$, differently from the one used to compute h, it is now defined as $\bar{B} = (y(0), \dot{y}(0), \ddot{y}(0), \cdots, y^{n-1}(0))$.

Finally, in order to obtain the complete solution y for (13) as shown in (15), we need to compute the particular solution "$y_p = (\sigma u) * h$", that is the convolution between the signal σu and the impulse response h. To avoid solving a convolution integral we can use the result of Theorem 3, by writing, if possible, the signal "σu" as a convolution (or a finite sum) of exponential signals of type "$\sigma(t)e^{rt}$", for some $r \in \mathbb{C}$. In this situation, as shown in examples in Section IV bellow, we increase the order of the Vandermonde matrix, as defined in Theorem 3, depending on how many "exponential modes" exists in the input signal "σu".

IV. Resolution of Differential Equations with Exponential Input Signals

In the following we show how to apply results discussed above to solve some specific differential equations:

Example IV-.1. Let be the second order initial value problem (IVP):

$$\ddot{y} + 3\dot{y} + 2y = 1, \quad \text{with } y(0) = -1 \text{ and } \dot{y}(0) = 2. \qquad (18)$$

To find the solution y, we consider the characteristic equation is $x^2 + 3x + 2 = 0$ whose roots are $r_1 = -1$ and $r_2 = -2$ and then $h_1(t) = \sigma(t)e^{-t}$ and $h_2 = \sigma(t)e^{-2t}$

(a) Impulse response (for $t > 0$): $h = h_1 * h_2 = A_1 h_1 + A_2 h_2$ and then $h(t) = A_1 e^{-t} + A_2 e^{-2t}$, where A_1 and A_2 are computed as

$$\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

which implies $h(t) = e^{-t} - e^{-2t}$, for $t > 0$.

(b) Homogeneous solution: $y_h(t) = B_1 e^{-t} + B_2 e^{-2t}$, where B_1 and B_2 are computed as:

$$\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \implies \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

which implies $y_h(t) = -e^{-2t}$.

(c) Particular solution (for $t > 0$): $y_p = (\sigma u) * h$, and $(\sigma u)(t) = 1.\sigma(t) = \sigma(t)e^{0t}$, then

$$y_p = (\sigma u) * h = h * (\sigma u) = h_1 * h_2 * h_3$$

where $h_1(t) = \sigma(t)e^{-t}, h_2(t) = \sigma(t)e^{-2t}$ and $h_3(t) = \sigma(t)e^{0t}$, or:

$$y_p(t) = C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{0t}$$

where C_1, C_2 and C_3 are compute as the solution of the "augmented" Vandermonde system:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 0 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0.5 \\ 0.5 \end{bmatrix}$$

which implies $y_p(t) = -e^{-t} + 0.5e^{-2t} + 0.5$, for $t > 0$.

Finally, the solution for the IVP (18) is $y = y_h + y_p$ or

$$y(t) = -e^{-t} - 0.5e^{-2t} + 0.5, \quad t \geq 0.$$

Example IV-.2. Let be the following IVP

$$\ddot{y} + 4y = t\cos 2t, \quad y(0) = -2, \quad \dot{y}(0) = 4 \qquad (19)$$

whose characteristic equation is $x^2 + 4 = 0$ which implies $r_1 = 2i$ and $r_2 = -2i$ or $h_1(t) = e^{2it}$ and $h_2(t) = e^{-2it}$.

(a) Impulse response: $h(t) = A_1 e^{2it} + A_2 e^{-2it}$, and

$$\begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -0.25i \\ 0.25i \end{bmatrix}$$

Then

$$h(t) = -025ie^{2it} + 0.25ie^{-2it} = 0.5\sin 2t.$$

(b) Homogeneous solution: $y_h(t) = B_1 e^{2it} + B_2 e^{-2it}$, and

$$\begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \implies \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} -1-i \\ -1+i \end{bmatrix}$$

Then

$$y_h(t) = (-1-i)e^{2it} + (-1+i)e^{-2it} = -2\cos 2t + 2\sin 2t$$

(c) Particular solution: $y_p = (\sigma u) * h$, and $u(t) = t\cos 2t = t(e^{2it} + e^{-2it})/2$, or:

$$u = 0.5(u_1 + u_2), \quad u_1(t) = te^{2it}, \quad u_2(t) = te^{-2it}$$

and so, $y_p = 0.5(\sigma u_1) * h + 0.5(\sigma u_2) * h$. Since $u_1(t) = te^{2it}$ and $u_2(t) = te^{-2it}$, we have

$$\begin{aligned} \sigma u_1 &= h_3 * h_3, \quad h_3(t) = \sigma(t)e^{2it} \\ \sigma u_2 &= h_4 * h_4, \quad h_4(t) = \sigma(t)e^{-2it} \end{aligned}$$

Therefore

$$(\sigma u_1) * h = h_1 * h_2 * h_3 * h_3 \text{ and } (\sigma u_2) * h = h_1 * h_2 * h_4 * h_4$$

where

$$h_1(t) = h_3(t) = \sigma(t)e^{2it}, \quad h_2(t) = h_4(t) = \sigma(t)e^{-2it}$$

and then

$$\begin{aligned} ((\sigma u_1) * h)(t) &= C_0 e^{-2it} + p(t)e^{2it}, \\ ((\sigma u_2) * h)(t) &= D_0 e^{2it} + q(t)e^{-2it}, \end{aligned}$$

where

$$p(t) = C_1 + C_2 t + C_3 t^2/2 \text{ and } q(t) = D_1 + D_2 t + D_3 t^2/2$$

So we have

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -2i & 2i & 1 & 0 \\ -4 & -4 & 4i & 1 \\ 8i & -8i & -12 & 6i \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} -0.015625i \\ 0.015625i \\ 0.0625 \\ -0.25i \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2i & -2i & 1 & 0 \\ -4 & -4 & -4i & 1 \\ -8i & 8i & -12 & -6i \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0.015625i \\ -0.015625i \\ 0.0625 \\ 0.25i \end{bmatrix}$$

Since $y_p = 0.5(\sigma u_1) * h + 0.5(\sigma u_2) * h$ we have, after regrouping the terms

$$y_p(t) = -0.03125 \sin 2t + 0.0625t \cos 2t + 0.125t^2 \sin 2t$$

and the solution $y = y_h + y_p$ will be given by

$$\begin{aligned} y(t) &= 1.96875 \sin 2t - 2\cos 2t + 0.0625t \cos 2t + \\ &+ 0.125t^2 \sin 2t, \quad t \geq 0 \end{aligned}$$

V. Conclusion

We showed in this paper a technique for computing the convolution of exponential signals, that avoids the need of solving integrals. The method is essentially algebraic and requires the resolution of Vandermonde systems, which is a well-known and extensively discussed problem in literature (see e.g. [8] and references therein). As discussed in [8, pp. 182], there exists efficient computational algorithms for solving Vandermod systems that gives accurate solution even when the Vandemond Matrix is ill-conditioned.[3] The proposed approach can be useful for the analysis of LTI systems in time domain as well as an alternative technique to solve ordinary differential equations with constant coefficients in a very simple way, and we believe it can be teached in a first course on linear systems theory. We note that the question of computing convolution of exponentials have been discussed previously in literature ([10], [11]), but the proposed approach is apparently different from the previous ones, and additionally is quite simple and suitable to be implemented computationally. Finally we note that the presented results can be extended to discrete-time context, where the exponential signal is now a function $f : \mathbb{Z} \to \mathbb{C}$, defined by $f(k) = a^k$, with $a \in \mathbb{C}$. Similar results can be obtained for the convolution of these signals as well as for the analysis of discrete time LTI systems and for the solution of difference equations (see [2]).

[3]Conditioning of Vandemond Matrix, as shown in (7), depends on position of its "nodes" r_i in complex plane (see [9] for a detailed discussion).

References

[1] Wikipedia: The Free Encyclopedia. Wikimedia Foundation, Inc. 21 Dec. 2017. Web. 21 Dec., 2017. Available at https://en.wikipedia.org/wiki/Convolution.

[2] F. Mota. *Computing the Convolution of Analog and Discrete Time Exponential Signals Algebraically*. arXiv:1606.08072v1, 2016.

[3] Scilab Enterprises. Scilab: Free and Open Source Software for Numerical Computation. Orsay, France, 2012. Available at http://www.scilab.org.

[4] W. Rudin. Functional Analysis, Second Edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Singapore, 1991.

[5] G. Grubb. *Distributions and Operators*. Graduate Texts in Mathematics (Book 252). Springer Science+Business Media, LLC, NY, 2009.

[6] K. L. Chung. *Elementary Probability Theory with Stochastic Processes*. Third Edition. Springer-Verlag NY Inc., 1979.

[7] Wikipedia: The Free Encyclopedia. Wikimedia Foundation, Inc. 4 June 2016. Web. 21 June, 2016. Available at http://en.wikipedia.org/wiki/Convolution_power.

[8] G. H. Golub and C. F. Van Loan. *Matrix Computations*. Second Edition. The Johns Hopkins Univ. Press, 1989.

[9] W. Gautschi. *How (Un)stable Are Vandermonde Systems?* In Asymptotic and Computational Analysis, pp. 193–210. Lectures Notes in Pure and Applied mathematics. R. Wong, Ed. Marcel Dekker, 1990.

[10] M. Akkouchi. *On The Convolution of Exponential Distributions*. Journal of The Chungcheong Mathematical Society, Vol. 21, No. 4, December 2008.

[11] N.-Y. Ma, F. Liu. *A Novel Analytical Scheme to Compute the n-Fold Convolution of Exponential-Sum Distribution Functions*. Applied Mathematics and Computation, 158 (2004) 225–235.

Appendix

1) Proof of Theorem 1: We note that $(h_1 * h_2 * \cdots * h_n)(0^+) = 0$ if $n \geq 2$, since this involves an integration of exponentials over an infinitesimal interval; this proves that $(h_1 * h_2 * \cdots * h_n)^{(0)}(0^+) = 0$. Now consider $(h_1 * h_2 * \cdots * h_n)^{(i)}$ for $1 \leq i \leq n-2$:

$$\begin{aligned} (h_1 * \cdots * h_n)^{(i)} &= (\dot{h}_1 * \cdots * \dot{h}_i) * \underbrace{(h_{i+1} * \cdots * h_n)}_{\text{at least two terms}} \\ &= \underbrace{[(\delta + r_1 h_1) * \cdots * (\delta + r_i h_i)]}_{\delta + f} * \\ &\quad * (h_{i+1} * \cdots * h_n) \\ &= (\delta + f) * (h_{i+1} * \cdots * h_n) \\ &= (h_{i+1} * \cdots * h_n) + \\ &\quad + f * (h_{i+1} * \cdots * h_n) \end{aligned}$$

Since the two terms in sum $(h_{i+1} * \cdots * h_n + f * h_{i+1} * \cdots * h_n)$ above are composed by a convolution of at least two signals, we conclude that $(h_1 * h_2 * \cdots * h_n)^{(i)}(0^+)$ is equals to zero. Now, considering $i = n-1$, we have:

$$\begin{aligned} (h_1 * \cdots * h_n)^{(n-1)} &= (\dot{h}_1 * \cdots * \dot{h}_{n-1}) * h_n \\ &= (\delta + r_1 h_1) * \cdots \\ &\quad * (\delta + r_{n-1} h_{n-1}) * h_n \\ &= (\delta + f) * h_n = \delta * h_n + f * h_n \\ &= h_n + f * h_n, \end{aligned} \tag{20}$$

and since $f * h_n$ is a sum of (at least) two signals convolution, we have that $(f * h_n)(0^+) = 0$ and consequently we obtain from (20):

$$(h_1 * h_2 * \cdots * h_n)^{(n-1)}(0^+) = (f * h_n)(0^+) + h_n(0^+) = 1$$

$\square$

Remark 1. The result of Theorem 1 can of course be extended to any set differentiable signals, other than exponentials. By considering $h_i = \sigma f_i$, where f_i is differentiable, we will get the following generalization of Equation (5):

$$(h_1 * h_2 * \cdots * h_n)^{(i)}(0^+) = \begin{cases} 0, & i = 0, 1, \ldots, n-2 \\ \prod_{j=1}^{n} f_j(0), & i = n-1 \end{cases}$$

2) Proof of Lemma 2.3: We prove by induction on (n_1, n_2). It is obviously true for $(n_1, n_2) = (1, 1)$.

1) Induction on n_1: Valid for $n_1 = k$ and $n_2 = 1$. Let it be $n_1 = k + 1$:

$$\begin{aligned} & h_1^{*(k+1)} * h_2 = h_1 * (h_1^{*k} * h_2) = \\ & h_1 * (A_1 h_1 + A_2 h_1^{*2} + \cdots A_k h_1^{*k} + B_1 h_2) = \\ & A_1 h_1^{*2} + A_2 h_1^{*3} + \cdots A_{k+1} h_1^{*(k+1)} + B_1 (h_1 * h_2) = \\ & A_1 h_1^{*2} + A_2 h_1^{*3} + \cdots A_{k+1} h_1^{*(k+1)} + B_1 (C_1 h_1 + C_2 h_2) = \\ & (B_1 C_1) h_1 + A_1 h_1^{*2} + A_2 h_1^{*3} + \cdots \\ & + A_{k+1} h_1^{*(k+1)} + (B_1 C_2) h_2 \end{aligned}$$

2) Induction on n_2: Valid for generic n_1 and $n_2 = k$. Let it be $n_2 = k+1$: Since $h_1^{*n_1} * h_2^{*(k+1)} = (h_1^{*n_1} * h_2^{*k}) * h_2$, then:

$$\begin{aligned} & (h_1^{*n_1} * h_2^{*k}) * h_2 = \\ & \big[(A_1 h_1 + A_2 h_1^{*2} + \cdots + A_{n_1} h_1^{*n_1}) + \\ & (B_1 h_2 + B_2 h_2^{*2} + \cdots + B_k h_2^{*k})\big] * h_2 = \\ & \underbrace{A_1 (h_1 * h_2) + A_2 (h_1^{*2} * h_2) + \cdots + A_{n_1} (h_1^{*n_1} * h_2)}_{\text{Rearranged as } (C_1 h_1 + C_2 h_1^{*2} + \cdots + C_{n_1} h_1^{*n_1} + D h_2)} + \\ & + B_1 h_2^{*2} + B_2 h_2^{*3} + \cdots + B_k h_2^{*(k+1)} = \\ & C_1 h_1 + C_2 h_1^{*2} + \cdots + C_{n_1} h_1^{*n_1} + \\ & + D h_2 + B_1 h_2^{*2} + B_2 h_2^{*3} + \cdots + B_k h_2^{*(k+1)} \end{aligned}$$

□

3) Proof of Theorem 3: We use induction on q to prove (10), which is valid for $q = 2$, as shown in Lemma 2.3. Suppose (10) is valid for $q = k$, and we prove it for $q = k+1$:

$$\begin{aligned} & h_1^{*n_1} * h_2^{*n_2} * \cdots * h_k^{*n_k} * h_{k+1}^{*n_{k+1}} = \\ & (h_1^{*n_1} * h_2^{*n_2} * \cdots * h_k^{*n_k}) * h_{k+1}^{*n_{k+1}} = \\ & \left(\sum_{j=1}^{n_1} A_{1j} h_1^{*j} + \sum_{j=1}^{n_2} A_{2j} h_2^{*j} + \cdots + \sum_{j=1}^{n_k} A_{kj} h_k^{*j} \right) * h_{k+1}^{*n_{k+1}} = \\ & \sum_{j=1}^{n_1} A_{1j} (h_1^{*j} * h_{k+1}^{*n_{k+1}}) + \sum_{j=1}^{n_2} A_{2j} (h_2^{*j} * h_{k+1}^{*n_{k+1}}) + \cdots \\ & + \sum_{j=1}^{n_k} A_{kj} (h_k^{*j} * h_{k+1}^{*n_{k+1}}) = \\ & \sum_{j=1}^{n_1} B_{1j} h_1^{*j} + \sum_{j=1}^{n_2} B_{2j} h_2^{*j} + \cdots + \sum_{j=1}^{n_k} B_{kj} h_k^{*j} + \\ & + \sum_{j=1}^{n_{k+1}} B_{(k+1)j} h_{k+1}^{*j} \end{aligned}$$

and the (10) is proved. To prove (12) we take the i-th derivative at $t = 0^+$ on both sides of (10) to get:

$$\begin{aligned} & (h_1^{*n_1} * h_2^{*n_2} * \cdots * h_q^{*n_q})^{(i)}(0^+) = \\ & \sum_{j=1}^{n_1} A_{1j} (h_1^{*j})^{(i)}(0^+) + \sum_{j=1}^{n_2} A_{2j} (h_2^{*j})^{(i)}(0^+) + \cdots \\ & + \sum_{j=1}^{n_q} A_{qj} (h_q^{*j})^{(i)}(0^+), \quad i = 0, 1, 2, \ldots, n-1. \end{aligned}$$

Applying Theorem 1 to left side of equation above and using Lemma 2.2, i.e., for $j \geq 1$:

$$(h_s^{*j})^{(i)}(0^+) = \binom{i}{j-1} r_s^{i-j+1}$$

we get (12). □