A Technique for Computing the Convolution of Exponential Signals and its Application in Systems Theory

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Abstract—We present a procedure for computing the convolution of (analog-time) exponential signals without the need of solving integrals. The procedure is algebraic and requires the resolution of a system of linear equations. The method can be applied to find the impulse response of a linear time-invariant (LTI) system modeled by an ordinary differential equation and also to compute the solution of an ordinary differential equation with constant coefficients. We believe this technique can be useful for the analysis of LTI systems in time domain, since it is conceptually simple and can be easily implemented as algorithms in software packages like Scilab and/or Matlab.

Keywords: Exponential Signal, Convolution, Vandermonde Matrix, Linear Time-Invariant (LTI) System, Ordinary Differential Equation.

I. INTRODUCTION

Convolution between signals [1] is a fundamental operation in the context of the theory of linear time invariant (LTI) systems and its importance comes mainly from the fact that this operation is linear on the signals and has the property of commuting with translations and differentiations. These properties imply the following pivotal result in theory of systems: The output signal of a LTI system is given by the convolution between the input signal and the system impulse response signal. So, the impulse response completely characterizes a LTI system for the purpose of input-output analysis. For LTI systems whose model of ordinary differential equation (ODE) is known, we have the important result that “the system impulse response can be decomposed as a convolution of exponential signals”, which can be obtained from the model [2]. These facts imply that the computation of convolution involving exponential signals is an important question in the context of linear systems theory and we consider it deserves a more deep analysis than is generally presented in basic textbooks on theory of linear systems. As we show below, convolution between exponential signals have some properties that can be explored to devise algorithms for obtaining a LTI impulse response as well as for solving ODEs in a very simple way. Due to its simplicity we believe that this approach may be taught even in a first course on linear systems theory.

A. Notation and Definitions

The notation we use is quite standard and it is presented here only for the sake of self-containment:

- \( \mathbb{Z}, \mathbb{R} \) and \( \mathbb{C} \) are, respectively, the set of integers, real and complex numbers;

- The binomial coefficient is defined for \( n, p \geq 0 \in \mathbb{Z} \) as:
  \[
  \binom{n}{p} = \begin{cases} 
  0, & n < p \\
  \frac{n!}{(n-p)!p!}, & n \geq p
  \end{cases}
  \]

- An (analog-time) signal is defined as a complex valued function \( f : \mathbb{R} \rightarrow \mathbb{C} \). In this paper we are mainly concerned with exponential signals, that is, \( f(t) = e^{rt} \) where \( r \in \mathbb{C} \). Two basic signals will be necessary in our development, namely, the unit step signal \( \sigma \) and the unit impulse (generalized) signal \((\delta)\). The unit step \( \sigma \) is defined as
  \[
  \sigma(t) = \begin{cases} 
  0, & t < 0 \\
  1, & t > 0
  \end{cases}
  \]
  and the unit impulse can be defined as the derivative of \( \sigma \), that is, \( \delta = \dot{\sigma} \). If \( f \) is a signal, the product \( \sigma f \) is given by
  \[
  (\sigma f)(t) = \sigma(t)f(t) = \begin{cases} 
  0, & t < 0 \\
  f(t), & t > 0
  \end{cases}
  \]
  If \( f \) is continuous at \( t = 0 \) we can also compute the product \( \delta f \) as \( \delta f = f(0)\delta \). Additionally, if the signal \( f \) is differentiable we can obtain the derivative of \( \sigma f \) as shown bellow:
  \[
  (\sigma f)' = \delta f + \sigma f = \delta f + \sigma \dot{f} = f(0)\delta + \sigma \dot{f},
  \]
  which will be useful in proof of Theorem 1.

- The convolution between two signals \( f \) and \( g \), represented by \( f \ast g \), is the binary operation defined as (see [1]):
  \[
  (f \ast g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau
  \] (1)

A generalized signal is in fact a distribution, as defined in [4], [5].
Additionally, if \( f(t) = g(t) = 0 \) for \( t < 0 \), we get from (1) that:

\[
(f * g)(t) = \begin{cases} 
0, & t < 0 \\
\int_0^t f(\tau)g(t-\tau)d\tau, & t > 0
\end{cases}
\]  

(2)

Convolution is commutative, associative and the unity of the operation is the (generalized) signal \( \delta \), that is, \( f * \delta = \delta * f = f \) for any signal \( f \). Another important property of convolution is related with signal derivative:

\[
(f * g) = f * g = f \ast \dot{g}
\]  

(3)

II. CONVOLUTION BETWEEN EXPONENTIAL SIGNALS

In this section we consider signals \( h : \mathbb{R} \to \mathbb{C} \) defined by:

\[
h(t) = \begin{cases} 
0, & t < 0 \\
e^{rt}, & t > 0 
\end{cases}, \quad r \in \mathbb{C},
\]  

(4)

or, alternatively, \( h(t) = \sigma(t)e^{rt} \), where \( \sigma \) is the unit step signal.

Theorem 1. Consider the convolution of \( n \geq 2 \) signals \( \langle h_1, h_2, \ldots, h_n \rangle \) with \( h_j(t) = \sigma(t)e^{r_jt} \), where \( r_j \in \mathbb{C} \) is the unit step signal and \( r_j \in \mathbb{C} \). The i-th derivative of the convolution \( (h_1 \ast h_2 \ast \cdots \ast h_n) \), represented by \( (h_1 \ast h_2 \ast \cdots \ast h_n)^{(i)} \), evaluated at \( t = 0^+ \) is given by:

\[
(h_1 \ast h_2 \ast \cdots \ast h_n)^{(i)}(0^+) = \begin{cases} 
0, & i = 0, 1, \ldots, n - 2 \\
1, & i = n - 1
\end{cases}
\]  

(5)

and we consider \( (h_1 \ast h_2 \ast \cdots \ast h_n)^{(0)} = h_1 \ast h_2 \ast \cdots \ast h_n \).

Proof. See Appendix.

So, vector \( A \) is the last (n-th) column of the inverse of \( V \).

Proof. We use induction on \( n \) to prove (6), which is easily verified for \( n = 2 \) using definition of convolution in Equation (2). Supposing (6) valid for \( n = k \geq 2 \), and using the fact that convolution is associative we easily prove (6) for \( n = k + 1 \). To obtain (7) we take the i-th derivative at \( t = 0^+ \) on both sides of (6) to get:

\[
(h_1 \ast \cdots \ast h_n)^{(i)}(0^+) = A_i h_1^{(i)}(0^+) + \cdots + A_n h_n^{(i)}(0^+),
\]

\( i = 0, 1, 2, \ldots, n - 1 \).

Applying Theorem 1 to left side of equation above and using the fact that \( h_j^{(i)}(0^+) = r_j^i \) we get (7).

Now we consider the more general convolution \( h_1 \ast h_2 \ast \cdots \ast h_n, \ n \geq 2 \), where there is the possibility of some \( h_i \) to be repeated, that is \( h_i = h_j \) for some \( i \neq j \). We initially consider some facts about the so-called “n-fold convolution” (or “convolution power” [6], [7]) of exponentials, that is, the convolution of \( h \), as defined in (4), repeated between itself \( n \) times, and we represent it by \( h^n \).

Lemma 2.1. The n-fold convolution of \( h(t) = \sigma(t)e^{rt} \), is given by

\[
h^n(t) = (h \ast h \ast \cdots \ast h)(t) = \frac{1}{(n-1)!} (t-n)^{n-1} h(t), \quad n \geq 1
\]  

(8)

Proof. By induction on \( n \). It is trivially true for \( n = 1 \) and supposing it valid for \( n = k \) we can prove for \( n = k + 1 \) by writing \( h^{(k+1)} = h^k \ast h \) and using definition of convolution in Equation (2).

The Lemma below shows a version of Theorem 1 applied to the n-fold convolution of \( h \):

Lemma 2.2. Let be \( h(t) = \sigma(t)e^{rt} \), then i-th derivative of \( h^n(t) \), \( n \geq 1 \), computed at \( t = 0^+ \) is given by:

\[
(h^n)^{(i)}(0^+) = \binom{i}{n-1} r^{i-n+1}
\]  

(9)

Proof. Equation (9) follows from Lemma 2.1 by setting \( k = n - 1 \) in the well-known formula below:

\[
\frac{d}{dt} \left( \frac{t^k}{k!} e^{rt} \right)_{t=0} = \binom{k}{i} r^{k-i}
\]

Now we analyse how it would be like the convolution \( h_1^{n_1} \ast h_2^{n_2} \), where \( h_1(t) = \sigma(t)e^{r_1t} \) and \( h_2(t) = \sigma(t)e^{r_2t} \), with \( r_1 \neq r_2 \), that is the convolution between the “n_1-fold” convolution of \( h_1 \) with the “n_2-fold” convolution of \( h_2 \) when \( h_1 \neq h_2 \).

Lemma 2.3. Let be \( h_1(t) = \sigma(t)e^{r_1t} \) and \( h_2(t) = \sigma(t)e^{r_2t} \), with \( r_1 \neq r_2 \), the convolution between the n_1-fold convolution
of $h_1$ and the $n_2$-fold convolution of $h_2$, denoted by $h_1^{*n_1} \ast h_2^{*n_2}$, is given by:

$$h_1^{*n_1} \ast h_2^{*n_2} = \left( h_1 \ast \left( h_1 \ast \cdots \ast h_1 \right) \right) \ast \left( h_2 \ast \left( h_2 \ast \cdots \ast h_2 \right) \right)$$

$$= \left( A_{11} h_1 + A_{12} h_2 + \cdots + A_{1n_1} h_1^{*n_1} \right) + \left( B_{11} h_2 + B_{12} h_2^{*2} + \cdots + B_{n_2} h_2^{*n_2} \right)$$

**Proof.** See Appendix.

Bellow we present a general result about the power convolution of $n$ exponential signals as shown in (4) which is a generalization of Theorem 2:

**Theorem 3.** The convolution between $n \geq 2$ exponential signals $\{h_1, h_2, \ldots, h_n\}$, with $h_1(t) = \sigma(t)e^{\gamma_1t}$, $r_i \in \mathbb{C}$ and $q$ distinct $h_q$, each one of them repeated $n_q$ times, so that $n_1 + n_2 + \cdots + n_q = n$, is given by

$$h_1^{*n_1} \ast h_2^{*n_2} \ast \cdots \ast h_q^{*n_q} = \sum_{j=1}^{n_1} A_{1j} h_1^{*j} + \sum_{j=1}^{n_2} A_{2j} h_2^{*j} + \cdots + \sum_{j=1}^{n_q} A_{nj} h_q^{*j}$$

$$= \left( A_{11} h_1 + A_{12} h_2 + \cdots + A_{1n_1} h_1^{*n_1} \right) + \left( B_{11} h_2 + B_{12} h_2^{*2} + \cdots + B_{n_2} h_2^{*n_2} \right) + \cdots + \left( D_{1q} h_q + D_{1q} h_q^{*q} \right).$$

Alternatively, by using (8) in Lemma 2.1 we have $h_i(t) = \sigma(t)e^{\gamma_i t}$, and then we can rewrite (10) as:

$$h_1^{*n_1} \ast h_2^{*n_2} \ast \cdots \ast h_q^{*n_q} = p_1 h_1 + p_2 h_2 + \cdots + p_q h_q$$

where each $p_s$, $s = 1, \ldots, q$, is a polynomial defined as

$$p_s(t) = \sum_{j=1}^{n_s} \sigma_q h_q^{*j}$$

and constants $A_{sj} \in \mathbb{C}$ are scalars that can be computed by solving a linear system $VA = B$ where $V$ is the $n \times n$ nonsingular conmutant (or generalized) Vandermonde matrix defined by $V = [V_1, V_2, \ldots, V_q]$, where each block $V_s$ is the $n \times n_s$ matrix whose entries are defined by

$$(V_s)_{ij} = \binom{i - 1}{j - 1} r_s^{i-j},$$

$A$ and $B$ are the $n$-column vectors $A = (A_1, A_2, \ldots, A_q)$, each $A_s$ is an $n_s$-column vector, and $B = (b_1, b_2, \ldots, b_q)$, where $b_0$ are $n_0$-column zero vectors and $B_q$ is the $n_q$-column vector $(0, 0, \cdots, 1)$ that is:

$$[V_1, V_2, \cdots, V_q] = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_q \end{bmatrix}$$

$$= \begin{bmatrix} 0_1 \\ 0_2 \\ \vdots \\ 0_{n_q} \end{bmatrix}$$

So, vector $A$ is the last (n-th) column of the inverse of $V$.

**Proof.** See Appendix.

### III. Analysis of LTI Systems Modeled by Ordinary Differential Equations

Consider a causal LTI system modeled by the following ordinary differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y + a_0y = u, \quad a_i \in \mathbb{R}$$

where $u$ is the input signal and $y$ is the output signal. The impulse response ($h$) for this system is given by the convolution:

$$h = h_1 \ast h_2 \ast \cdots \ast h_n, \quad h_i(t) = \sigma(t)e^{\gamma_i t}, \quad r_i \in \mathbb{C}$$

and $r_1, r_2, \ldots, r_n$ are the roots of the characteristic equation $x^n + a_{n-2}x^{n-1} + \cdots + a_1x + a_0 = 0$. Supposing that the characteristic equation has $q$ distinct roots $r_s$, each one repeated $n_s$ times, so that $n_1 + n_2 + \cdots + n_q = n$, then we can obtain the impulse response $h$ by using Theorem 3, Equation (11), that is

$$h = p_1 h_1 + p_2 h_2 + \cdots + p_q h_q$$

where $h_s(t) = \sigma(t)e^{\gamma_s t}$, $p_s(t) = \sum_{j=1}^{n_s} \sigma_q h_q^{*j}$, and $A_{sj}$, $s = 1, \ldots, q$, $j = 1, \ldots, n_s$, are calculated by solving the Vandermonde system (12).

On the other hand, it is well known that the complete solution of differential equation (13) is generally written as

$$y = y_h + y_p$$

where $y_h$ is the homogeneous (or zero input) solution and $y_p$ is a particular solution, i.e., it depends on input signal $u$. When solving (13) for $t \geq 0$, the particular solution $y_p$ can be written as

$$y_p(t) = \int_0^t u(\tau)h(t-\tau)d\tau = \langle \sigma u \rangle h(t),$$

where $\langle \sigma u \rangle(t) = \begin{cases} 0, & t < 0 \\ u(t), & t \geq 0 \end{cases}$

and $h$ is the impulse response. The homogeneous solution ($y_h$) has the same format of (14), that is

$$y_h = \mathbf{p}_1 h_1 + \mathbf{p}_2 h_2 + \cdots + \mathbf{p}_q h_q,$$

where $h_s(t) = e^{\gamma_s t}$, and $\mathbf{p}_s(t) = \sum_{j=1}^{n_s} \sigma_q h_q^{*j}$.

Therefore to solve (13) we need to obtain $y_p$ and $y_h$. To find $y_h$ we use the fact that the particular solution $y_p$, as shown in (16), is a convolution between $n + 1$ signals, namely, “$y_p = \langle \sigma u \rangle * h \ast h_1 \ast h_2 \ast \cdots \ast h_n$”, and conclude, by using Theorem 1, that:

$$y_p(0^+) = \tilde{y}_p(0^+) = \check{y}_p(0^+) = \cdots = y_p^{(n-1)}(0^+) = 0$$

and so, using these conditions in (15), we get:

$$y(0^+) = y_h(0^+), \quad \check{y}(0^+) = y_h(0^+), \quad \tilde{y}(0^+) = y_h(0^+), \cdots,$$

$$y^{(n-1)}(0^+) = y_h^{(n-1)}(0^+).$$

This set of conditions on $y_h$ can be used to find the constants $\mathbf{A}_{sj}$ in (17) since the “initial values”
\( y(0), \dot{y}(0), \ddot{y}(0), \ldots, y^{(n-1)}(0) \) are generally known when solving (13) for \( t \geq 0 \). This implies that the constants \( A_{1,j}, s = 1, \ldots, q \) and \( j = 1, \ldots, n_s \), can be computed by solving a Vandermonde system like the one showed in Theorem 3, that is \( V \tilde{A} = \tilde{B} \), where the Vandermonde matrix \( V \) is the same one used to compute the impulse response \( h \), \( \tilde{A} \) is the \( n \times 1 \) vector composed by the \( A_{1,j} \)'s and the vector \( \tilde{B} \), differently from the one used to compute \( h \), it is now defined as \( \tilde{B} = (y(0), \dot{y}(0), \ddot{y}(0), \ldots, y^{(n-1)}(0)) \).

Finally, in order to obtain the complete solution \( y \) for (13) as shown in (15), we need to compute the particular solution \( y_p = (\sigma u) \ast h \), that is the convolution between the signal \( \sigma u \) and the impulse response \( h \). To avoid solving a convolution integral we can use the result of Theorem 3, by writing, if possible, the signal \( \sigma u \) as a convolution (or a finite sum) of exponential signals of type \( \sigma(t)e^{rt} \), for some \( r \in \mathbb{C} \).

In this situation, as shown in Examples 4.1 below, we increase the order of the Vandermonde matrix, as defined in Theorem 3, depending on how many "exponential modes" exists in the input signal \( \sigma u \).

IV. Resolution of Differential Equations with Exponential Input Signals

In the following we show how to apply results discussed above to solve some specific differential equations:

**Example IV.1.** Let be the second order initial value problem (IVP):

\[ \ddot{y} + 3\dot{y} + 2y = 1, \quad \text{with} \ y(0) = -1 \quad \text{and} \ y(0) = 2. \]  

(18)

To find the solution \( y \), we consider the characteristic equation \( x^2 + 3x + 2 = 0 \) whose roots are \( r_1 = -1 \) and \( r_2 = -2 \) and then \( h_1(t) = e^{-t} \) and \( h_2(t) = e^{-2t} \).

(a) Impulse response (for \( t > 0 \)): \( h = h_1 \ast h_2 = A_1h_1 + A_2h_2 \) and then \( h(t) = A_1e^{-t} + A_2e^{-2t} \), where \( A_1 \) and \( A_2 \) are computed as

\[
\begin{bmatrix}
1 & 1 \\
-1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
\end{bmatrix} \implies A_1 = 1, \quad A_2 = 2
\]

which implies \( h(t) = e^{-t} - e^{-2t} \), for \( t > 0 \).

(b) Homogeneous solution: \( y_h(t) = B_1e^{-t} + B_2e^{-2t} \), where \( B_1 \) and \( B_2 \) are computed as:

\[
\begin{bmatrix}
1 & 1 \\
-1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
\end{bmatrix} = \begin{bmatrix}
y(0) \\
\dot{y}(0) \\
\end{bmatrix} = \begin{bmatrix}
-1 \\
2 \\
\end{bmatrix} \implies B_1 = 0, \quad B_2 = -1
\]

which implies \( y_h(t) = -e^{-2t} \).

(c) Particular solution (for \( t > 0 \)):

\( y_p = (\sigma u) \ast h \), and \( \sigma(t) = \sigma(t)e^{0t} \), then

\( y_p(t) = (\sigma u) \ast h = h \ast (\sigma u) = h_1 \ast h_2 \ast h_3 \)

where \( h_1(t) = \sigma(t)e^{-t} \), \( h_2(t) = \sigma(t)e^{-2t} \) and \( h_3(t) = \sigma(t)e^{0t} \), or:

\( y_p(t) = C_1e^{-t} + C_2e^{-2t} + C_3e^{0t} \)

where \( C_1, C_2 \) and \( C_3 \) are computed as the solution of the "augmented" Vandermonde system:

\[
\begin{bmatrix}
1 & 1 & 1 \\
-1 & -2 & 0 \\
1 & 4 & 0 \\
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix} \implies \begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
\end{bmatrix} = \begin{bmatrix}
-1 \\
0.5 \\
0.5 \\
\end{bmatrix}
\]

which implies \( y_p(t) = -e^{-t} + 0.5e^{-2t} + 0.5 \), for \( t > 0 \).

Finally, the solution for the IVP (18) is \( y = y_h + y_p \) or

\( y(t) = -e^{-t} - 0.5e^{-2t} + 0.5 \), \( \quad t > 0 \).

**Example IV.2.** Let be the following IVP

\( \ddot{y} + 4y = t \cos 2t, \quad y(0) = -2, \quad \dot{y}(0) = 4 \)  

(19)

whose characteristic equation is \( x^2 + 4 = 0 \) which implies \( r_1 = 2i \) and \( r_2 = -2i \) or \( h_1(t) = e^{2it} \) and \( h_2(t) = e^{-2it} \).

(a) Impulse response: \( h(t) = A_1e^{2it} + A_2e^{-2it} \), and

\[
\begin{bmatrix}
1 & 1 \\
2i & -2i \\
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
\end{bmatrix} \implies A_1 = 0.25i, \quad A_2 = -0.25i
\]

Then

\( h(t) = -0.25ie^{2it} + 0.25ie^{-2it} = 0.5 \sin 2t \).

(b) Homogeneous solution: \( y_h(t) = B_1e^{2it} + B_2e^{-2it} \), and

\[
\begin{bmatrix}
1 & 1 \\
2i & -2i \\
\end{bmatrix}
\begin{bmatrix}
B_1 \\
B_2 \\
\end{bmatrix} = \begin{bmatrix}
-2 \\
4 \\
\end{bmatrix} \implies B_1 = -1 - i, \quad B_2 = -1 + i
\]

Then

\( y_h(t) = (-1 - i)e^{2it} + (-1 + i)e^{-2it} = -2 \cos 2t + 2 \sin 2t \).

(c) Particular solution: \( y_p = (\sigma u) \ast h \), and \( u(t) = t \cos 2t = t(e^{2it} + e^{-2it})/2 \), or:

\( u = 0.5(u_1 + u_2), \quad u_1(t) = te^{2it}, \quad u_2(t) = te^{-2it} \)

and so, \( y_p = 0.5(\sigma u_1) \ast h + 0.5(\sigma u_2) \ast h \). Since \( u_1(t) = te^{2it} \) and \( u_2(t) = te^{-2it} \), we have

\[ \sigma u_1 = h_3 \ast h_3, \quad h_3(t) = \sigma(t)e^{2it}, \]

\[ \sigma u_2 = h_4 \ast h_4, \quad h_4(t) = \sigma(t)e^{-2it} \]

Therefore

\( (\sigma u_1) \ast h = h_1 \ast h_2 \ast h_3 \ast h_3 \) and \( (\sigma u_2) \ast h = h_1 \ast h_2 \ast h_4 \ast h_4 \)

where

\( h_1(t) = h_3(t) = \sigma(t)e^{2it}, \quad h_2(t) = h_4(t) = \sigma(t)e^{-2it} \)

and then

\( (\sigma u_1) \ast h(t) = C_0e^{-2it} + p(t)e^{2it}, \quad (\sigma u_2) \ast h(t) = D_0e^{2it} + q(t)e^{-2it}, \)

where

\( p(t) = C_1 + C_2t + C_3t^2/2 \) and \( q(t) = D_1 + D_2t + D_3t^2/2 \)

\[ 101 \]
So we have

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
-2i & 2i & 1 & 0 \\
-4 & -4 & 4i & 1 \\
8i & -8i & -12 & 6i \\
\end{bmatrix}
\begin{bmatrix}
C_0 \\
C_1 \\
C_2 \\
C_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
C_0 \\
C_1 \\
C_2 \\
C_3 \\
\end{bmatrix}
= \begin{bmatrix}
-0.015625i \\
0.015625i \\
0.0625 \\
-0.25i \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
2i & -2i & 1 & 0 \\
-4 & -4 & -4i & 1 \\
8i & 8i & -12 & -6i \\
\end{bmatrix}
\begin{bmatrix}
D_0 \\
D_1 \\
D_2 \\
D_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
D_0 \\
D_1 \\
D_2 \\
D_3 \\
\end{bmatrix}
= \begin{bmatrix}
-0.015625i \\
0.015625i \\
0.0625 \\
0.25i \\
\end{bmatrix}
\]

Since \( y_p = 0.5(\sigma u_1) * h + 0.5(\sigma u_2) * h \) we have, after regrouping the terms

\[
y_p(t) = -0.03125 \sin 2t + 0.0625t \cos 2t + 0.125t^2 \sin 2t
\]

and the solution \( y = y_h + y_p \) will be given by

\[
y(t) = 1.96875 \sin 2t - 2 \cos 2t + 0.0625t \cos 2t + 0.125t^2 \sin 2t, \quad t \geq 0
\]

V. CONCLUSION

We showed in this paper a technique for computing the convolution of exponential signals, that avoids the need of solving integrals. The method is essentially algebraic and requires the resolution of Vandermonde systems, which is a well-known and extensively discussed problem in literature (see e.g. [8] and references therein). As discussed in [8, pp. 182], there exists efficient computational algorithms for solving Vandermonde systems that gives accurate solution even when the Vandermonde Matrix is ill-conditioned. The proposed approach can be useful for the analysis of LTI systems in time domain as well as an alternative technique to solve ordinary differential equations with constant coefficients in a very simple way, and we believe it can be taught in a first course on linear systems theory. We note that the question of computing convolution of exponential signals have been discussed previously in literature ([10], [11]), but the proposed approach is apparently different from the previous ones, and additionally is quite simple and suitable to be implemented computationally. Finally we note that the presented results can be extended to discrete-time context, where the exponential signal is now a function \( f : Z \rightarrow C \), defined by \( f(k) = a^k \), with \( a \in C \). Similar results can be obtained for the convolution of these signals as well as for the analysis of discrete time LTI systems and for the solution of difference equations (see [2]).

\[ \text{REFERENCES} \]


\[ \text{APPENDIX} \]

1) Proof of Theorem 1: We note that \( (h_1 * h_2 * \ldots * h_n)(0^+) = 0 \) if \( n \leq 2 \), since this involves an integration of exponentials over an infinitesimal interval; this proves that \( (h_1 * h_2 * \ldots * h_n)(0^+) = 0 \). Now consider \( (h_1 * h_2 * \ldots * h_n)(i) \) for \( 1 \leq i \leq n - 2 \):

\[
(h_1 * \ldots * h_n)^{(i)} = (h_{i+1} * \ldots * h_n)\left(\delta + r_i h_i\right) = \left(\delta + r_i h_i\right) * (h_{i+1} * \ldots * h_n)
\]

Since the two terms in sum \( (h_{i-1} * \ldots * h_n) + f * h_{i-1} * \ldots * h_n \) above are composed by a convolution of at least two signals, we conclude that \( (h_1 * h_2 * \ldots * h_n)(0^+) \) is equals to zero.

Now, considering \( i = n - 1 \), we have:

\[
(h_1 * \ldots * h_n)^{(n-1)} = (h_{n-1} * h_n)
= (\delta + r_{n-1} h_{n-1}) * \ldots
\]

and since \( f * h_n \) is a sum of (at least) two signals convolution, we have that \( (f * h_n)(0^+) = 0 \) and consequently we obtain from (20):

\[
(h_1 * h_2 * \ldots * h_n)^{(n-1)}(0^+) = (f * h_n)(0^+) + h_n(0^+) = 1
\]

\[ \square \]
Remark 1. The result of Theorem 1 can of course be extended to any set differentiable signals, other than exponentials. By considering \( h_i = \sigma_j f_i \), where \( f_i \) is differentiable, we will get the following generalization of Equation (5):

\[
(h_1 * h_2 * \cdots * h_n)^{(i)}(0^+) = \begin{cases} 
0, & i = 0, 1, \ldots, n - 2 \\
\prod_{j=1}^{n} f_j(0), & i = n - 1
\end{cases}
\]

2) Proof of Lemma 2.3: We prove by induction on \((n_1, n_2)\). It is obviously true for \((n_1, n_2) = (1, 1)\).

1) Induction on \(n_1\): Valid for \(n_1 = k\) and \(n_2 = 1\). Let it be \(n_1 = k + 1\):

\[
\begin{align*}
(h_1^{(k+1)} * h_2) &= h_1 * (h_1^{(k)} * h_2) = \\
&= h_1 * (A_1 h_3 + A_2 h_3^2 + \cdots + A_k h_3^k + B_1 h_2) = \\
&= A_1 h_3^2 + A_2 h_3^3 + \cdots + A_k h_3^{(k+1)} + B_1 (h_1 * h_2) = \text{we get (12)}, \\
&= (B_1 C_1) h_1 + (B_1 C_2) h_2 + \cdots + (B_1 C_k) h_2
\end{align*}
\]

Applying Theorem 1 to the left side of equation above and using Lemma 2.2, i.e., for \(j \geq 1\):

\[
(h_1^{(j)})(0^+) = \left( \frac{i}{j-1} \right) r^i_{j-1}
\]

2) Induction on \(n_2\): Valid for generic \(n_1\) and \(n_2 = k\). Let it be \(n_2 = k + 1\). Since \(h_1^{(n_1)} * h_2^{(k+1)} = (h_1^{(n_1)} * h_2^k) * h_2^k\), then:

\[
(h_1^{(n_1)} * h_2^k) * h_2 = \\
[(A_1 h_3 + A_2 h_3^2 + \cdots + A_n h_3^{n_1}) + (B_1 h_2 + B_2 h_2^2 + \cdots + B_k h_2^k)] * h_2 = \\
A_1 (h_1 * h_2) + A_2 (h_2^2 * h_2) + \cdots + A_n (h_2^{n_1} * h_2) + \\
\text{Rearranged as } (C_1 h_1 + C_2 h_2^2 + \cdots + C_n h_2^{n_1} + D h_2) \\
+ B_1 h_2^2 + B_2 h_2^3 + \cdots + B_k h_2^{(k+1)} = \\
C_1 h_1 + C_2 h_2^2 + \cdots + C_n h_2^{n_1} + \\
D h_2 + B_1 h_2^2 + B_2 h_2^3 + \cdots + B_k h_2^{(k+1)}
\]

3) Proof of Theorem 3: We use induction on \(q\) to prove (10), which is valid for \(q = 2\), as shown in Lemma 2.3. Suppose (10) is valid for \(q = k\), and we prove it for \(q = k + 1\):

\[
\begin{align*}
&h_1^{(n_1)} * h_2^{(n_2)} * \cdots * h_k^{(n_k)} * h_{k+1}^{(n_{k+1})} = \\
&(h_1^{(n_1)} * h_2^{(n_2)} * \cdots * h_k^{(n_k)} * h_{k+1}^{(n_{k+1})}) = \\
&\left(\sum_{j=1}^{n_1} A_{1j} h_1^{(j)} + \sum_{j=1}^{n_2} A_{2j} h_2^{(j)} + \cdots + \sum_{j=1}^{n_k} A_{kj} h_k^{(j)}\right) * h_{k+1}^{(n_{k+1})} = \\
&\sum_{j=1}^{n_1} A_{1j} (h_1^{(j)} * h_{k+1}^{(n_{k+1})}) + \sum_{j=1}^{n_2} A_{2j} (h_2^{(j)} * h_{k+1}^{(n_{k+1})}) + \cdots \\
&+ \sum_{j=1}^{n_k} A_{kj} (h_k^{(j)} * h_{k+1}^{(n_{k+1})}) = \\
&\sum_{j=1}^{n_1} B_{1j} h_1^{(j)} + \sum_{j=1}^{n_2} B_{2j} h_2^{(j)} + \cdots + \sum_{j=1}^{n_k} B_{kj} h_k^{(j)} + \\
&+ \sum_{j=1}^{n_{k+1}} B_{(k+1)j} h_{k+1}^{(j)}
\end{align*}
\]